Economics 2450A: Public Economics
Section 3-4: Mirrlees Taxation

Matteo Paradisi

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In today's section we will solve the Mirrlees tax problem. We will and derive optimal taxes introducing the concept of wedges and study the model with and without income effects.

1 The Model Setup

Suppose the agent has preferences over consumption and labor represented by the utility function $u(c,l)$ that we assume separable and quasi-linear such that $u(c,l) = c - v(l)$. We assume that $v'(l) > 0$ and $v''(l) \geq 0$. Each agent earns income $z = nl$ when supplying l hours of labor and consumes $c = nl - T(nl)$ after taxes. Individuals are heterogeneous in the salary n that represents their type and we will interpret as a measure of ability. Salaries are distributed according to $f(n)$, with $n \in [\underline{n}, \bar{n}]$. Individual welfare is aggregated through a social welfare function $G(\cdot)$, that we assume differentiable and concave.

Revelation Principle Throughout all of the tax problems that we study we will assume that the government cannot observe the labor choice of the agent and her type. Income is the only observed choice that the government can target. We solve the model using a revelation mechanism. Our goal is to define an optimal tax schedule that delivers an allocation $z(n)$, $c(n)$ to each agent n. The *Revelation Principle* claims that if an allocation can be implemented through some mechanism, then it can also be implemented through a direct truthful mechanism where the agent reveals her information about n.

We imagine that each agent reports to the government her type n' and that allocations are a function of n' such that we can write $c(n')$, $l(n')$, $z(n')$ and $u(n')$. By revelation principle, the government cannot do better than defining functions $c(n)$, $z(n)$ such that the agent finds optimal to reveal her true productivity:

$$c(n) - v\left(\frac{z(n)}{n}\right) \geq c(n') - v\left(\frac{z(n')}{n}\right)$$

for every n and n' where n is the true type of the agent. Notice that since n is continuous we have an infinity of constraints. In order to reduce the dimensionality of the problem, we assume that the marginal rate of substitution between consumption and before-tax income is decreasing in n:

$$-MRS_{cz} = \frac{v'(z(n)/n)}{nu'(c(n))} \text{ decreases in } n$$

This is the so called *single-crossing condition* (or *Spence-Mirrlees condition*). Single-crossing and incentive compatibility imply the monotonicity of allocations (i.e. $c(n)$, $z(n)$ are increasing in n). If monotonicity and single-crossing are satisfied, we can replace the incentive constraint with the first-order necessary conditions of the agent that provide a local incentive condition. Under monotonicity and single-crossing the local conditions are also sufficient. While solving these problems we will only impose local incentive constraints and ignore the monotonicity of allocations, which is then verified ex-post.

Incentive Compatibility We reduce the dimensionality of the problem by taking a *first order approach* that replaces the infinity of constraints for each individual with a local condition relying on the optimal revelation choice. When reporting, the individual of type n solves the following problem:

$$\max_{n'} c\left(n'\right) - v\left(\frac{z\left(n'\right)}{n}\right)$$

the first order necessary condition for this problem is:

$$c'\left(n'\right) - \frac{z'\left(n'\right)}{n} v'\left(\frac{z\left(n'\right)}{n}\right) = 0$$

If the government wants the agent to reveal her true type, it must be:

$$c'\left(n\right) = \frac{z'\left(n\right)}{n} v'\left(\frac{z\left(n\right)}{n}\right)$$

Under the concavity assumption on the preferences, this is a global incentive constraint condition. Suppose we study local utility changes by totally differentiating the utility wrt n, we get:

$$\frac{du\left(n\right)}{dn} = \left(c'\left(n\right) - \frac{z'\left(n\right)}{n} v'\left(\frac{z\left(n\right)}{n}\right)\right) + \frac{z\left(n\right)}{n^2} v'\left(\frac{z\left(n\right)}{n}\right)$$

Notice that the term in the first bracket is the first order condition of the agent. We can thus write $du\left(n\right)/dn = \left(z\left(n\right)/n^2\right) v'\left(z\left(n\right)/n\right)$. This equation pins down the slope of the utility assigned to the agent at the optimum. By convexity of $v\left(\cdot\right)$, the slope is always positive: the government assigns higher utility to higher types at the optimum. Higher types have a lower marginal disutility of labor for a given level of hours worked and they get *informational rents* in the equilibrium.

Labor Supply and Labor Wedge The individual solves the following optimization problem:

$$\max_{z} z - T\left(z\right) - v\left(\frac{z}{n}\right)$$

The first order condition is:

$$T'\left(z\right) = 1 - \frac{v'\left(l\right)}{n}$$

The second term on the right-hand-side of the equation is the marginal rate of substitution between consumption and income and we can always write that $T'\left(n\right) = 1 - MRS\left(n\right)$. When the agent is not distorted, the MRS is equal to 1 implying $T'\left(z\right) = 0$. We can interpret $T'\left(z\right)$ as a *wedge* on the optimal labor supply: whenever it is different from zero, labor supply is distorted. Wedges are a central concept in the optimal taxation literature and we will encounter them throughout the class.

From the optimality condition, we can derive the elasticity of labor wrt the net of tax wage. Rewrite the optimality condition as:

$$v'\left(\frac{z}{n}\right) = \left(1 - T'\left(z\right)\right) n$$

Totally differentiating wrt $\left(1 - T'\left(z\right)\right) n$, we have:

$$\frac{dl}{d\left(1 - T'\left(z\right)\right) n} v''\left(l\right) = 1$$

Which implies the following elasticity:

$$\varepsilon = \frac{dl}{d\left(1 - T'\left(z\right)\right) n} \frac{\left(1 - T'\left(z\right)\right) n}{l} = \frac{v'\left(z\right)}{l v''\left(z\right)}$$

Resource Constraint Suppose the government has an exogenous revenue requirement E. The revenues collected through taxation must be at least equal to E. Using the agent's budget constraint we can write the tax levied on a single individual as $T(z(n)) = z(n) - c(n)$. Summing over all the individuals in the economy we get:

$$\int_{\underline{n}}^{\bar{n}} c(n) f(n) dn \geq \int_{\underline{n}}^{\bar{n}} z(n) f(n) dn - E$$

This is the resource constraint for this economy. Notice that unlike incentive constraint, this constraint is unique.

2 Optimal Income Tax

We now solve the constrained maximization problem using optimal control theory. Instead of having taxes as a choice variable, we assume that the government chooses an allocation for each agent. Given the individual's budget constraint, this is equivalent to choosing a tax level. The government problem is:

$$\max_{c(n), u(n), z(n)} \int_{\underline{n}}^{\bar{n}} G(u(n)) f(n)$$

s.t.

$$\frac{du(n)}{dn} = \frac{z(n)}{n^2} v' \left(\frac{z(n)}{n} \right)$$

$$\int_{\underline{n}}^{\bar{n}} c(n) f(n) dn \geq \int_{\underline{n}}^{\bar{n}} z(n) f(n) dn - E$$

We solve the problem with a Hamiltonian where we interpret n as the continuous variable and choose $u(n)$ as state variable and $z(n)$ as control. The incentive constraint becomes the law of motion of the state variable: it measures how utility changes across types in equilibrium. In order to setup the Hamiltonian, we need to replace consumption in the resource constraint with a function of state and control variables. Using the definition of indirect utility, we can write $c(n) = u(n) + v(z(n)/n)$. We replace this condition into the resource constraint and setup the following Hamiltonian:

$$H = \left[G(u(n)) + \lambda \left(z(n) - u(n) - v \left(\frac{z(n)}{n} \right) \right) \right] f(n) + \mu(n) \frac{z(n)}{n^2} v' \left(\frac{z(n)}{n} \right)$$

$\mu(n)$ denotes the multiplier on the incentive constraint of type n and λ is the multiplier on the resource constraint.

The first order conditions of the optimal control problem are:

$$\frac{\partial H}{\partial z(n)} = \lambda \left[1 - \frac{v'(l(n))}{n} \right] f(n) + \frac{\mu(n)}{n^2} \left[v' \left(\frac{z(n)}{n} \right) + \frac{z(n)}{n} v'' \left(\frac{z(n)}{n} \right) \right] = 0 \quad (1)$$

$$\frac{\partial H}{\partial u(n)} = [G'(u(n)) - \lambda] f(n) = -\mu'(n) \quad (2)$$

The transversality (boundary) conditions read:

$$\mu(\underline{n}) = \mu(\bar{n}) = 0$$

The Hamiltonian solution requires $\mu(\bar{n}) u(\bar{n}) = 0$. However, if we want to provide positive utility to type $\bar{n}$ we must require $\mu(\bar{n}) = 0$. At the same time, since at the optimum the incentive constraints will be binding downwards, we require $\mu\left(\underline{n}\right)=0$. As it is standard in this kind of problems the lowest type does not want to imitate any other agent in equilibrium implying that her incentive constraint is slack, while everyone else is indifferent between her allocation and the allocation provided to the immediately lower type.

If we integrate equation (2) over the entire type space and use transversality conditions we find:

$$\lambda=\int_{\underline{n}}^{\bar{n}} G'\left(u\left(n\right)\right)f\left(n\right)dn$$

This is an expression for the marginal value of public funds to the government. It states that the value of public funds depends on the marginal social welfare gains across the entire type space and it is equal to the welfare effect of transferring $1 to every individual in the economy. In other words, public funds are more valuable the higher are the social welfare gains achievable in the economy.

We can also integrate equation (2) to find the value of $\mu\left(n\right)$:

$$-\mu\left(n\right)=\int_{n}^{\bar{n}}\left[\lambda-G'\left(u\left(m\right)\right)\right]f\left(m\right)dm \tag{3}$$

Using the definition of labor elasticity, we rearrange the following:

$$\left[v'\left(\frac{z\left(n\right)}{n}\right)+\frac{z\left(n\right)}{n}v''\left(\frac{z\left(n\right)}{n}\right)\right]=v'\left(\frac{z\left(n\right)}{n}\right)\left[1+\frac{1}{\epsilon}\right]$$

Exploiting the definition of the tax wedge, we simplify equation (1) to get:

$$\lambda T'\left(z\left(n\right)\right)=\frac{\mu\left(n\right)}{f\left(n\right)}\left(1-T'\left(z\left(n\right)\right)\right)\left(1+\frac{1}{\epsilon}\right)$$

Using the expression for μ derived in equation (3) we get:

$$\frac{T'\left(z\left(n\right)\right)}{1-T'\left(z\left(n\right)\right)}=\left(\frac{1+\epsilon}{\epsilon}\right)\frac{\int_{n}^{\bar{n}}\left[1-g\left(m\right)\right]f\left(m\right)dm}{nf\left(n\right)} \tag{4}$$

The optimal tax is decreasing in the elasticity of labor supply. We define $g\left(n\right)=\frac{G'\left(u\left(n\right)\right)}{\lambda}$ the relative social welfare weight of individual n such that $\int_{\underline{n}}^{\bar{n}} g\left(n\right)f\left(n\right)dn=1$. Remember that λ aggregates the social welfare weights across the entire economy. Thus, a higher $g\left(n\right)$ means that the government cares relatively more about individual n and will tax her less.

A Rawlsian government would have $g\left(n\right)=0$ for any $n>\underline{n}$ and the formula would reduce to:

$$\frac{T'\left(z\left(n\right)\right)}{1-T'\left(z\left(n\right)\right)}=\left(\frac{1+\epsilon}{\epsilon}\right)\frac{1-F\left(n\right)}{nf\left(n\right)}$$

The second part of the expression captures the ratio of the mass above type n and the density at n. It is a measure of thickness and the lower it is the higher marginal tax rate will be.

3 Diamond ABC Formula

In this paragraph we derive a tax formula presented in Diamond (1998). We change our assumption about welfare weights and assume that they are distributed according to a function $\psi\left(n\right)$ with cdf $\Psi\left(n\right)$. The government objective function becomes:

$$\int_{\underline{n}}^{\bar{n}} u\left(n\right)\psi\left(n\right)dn$$

By assumption $\int_{\underline{n}}^{\bar{n}} \psi(n)\,dn = 1$ implies $\lambda = 1$. First order conditions can be derived exactly as before. We therefore have:

$$-\mu'(n) = \psi(n) - \lambda f(n)$$

and after integration:

$$\begin{aligned} -\mu(n) &= \int_n^{\bar{n}} \left(f(n) - \psi(n)\right) dn \\ &= \Psi(n) - F(n) \end{aligned}$$

Using the expression above the tax formula reads:

$$\frac{T'(z(n))}{1 - T'(z(n))} = \left(\frac{1+\epsilon}{\epsilon}\right) \frac{\Psi(n) - F(n)}{nf(n)}$$

To write the ABC formula we divide and multiply by $1 - F(n)$ to get:

$$\frac{T'(z(n))}{1 - T'(z(n))} = \underbrace{\left(\frac{1+\epsilon}{\epsilon}\right)}_{A(n)} \underbrace{\frac{\Psi(n) - F(n)}{1 - F(n)}}_{B(n)} \underbrace{\frac{1 - F(n)}{nf(n)}}_{C(n)}$$

$A(n)$ captures the standard elasticity and efficiency argument. $B(n)$ measures the desire for redistribution: if the sum of weights below n is high relative to the mass above n, the government will tax more. Finally, $C(n)$ measures the thickness of the right tail of the distribution. A thicker tail will be associated to higher tax rates.

Notice that in the Rawlsian case $\Psi(n) = 1$ for every $n > \underline{n}$ and the formula converges to the one presented in the previous paragraph.

4 Optimal Taxes With Income Effects

We now relax the assumption of no income effects. Suppose the utility of the agent takes the form $\tilde{u}(c, l) = u(c) - v(l)$ where $u'(c) > 0$ and $u''(c) \leq 0$.

Elasticity of Labor Supply The optimality condition for the labor supply choice becomes:

$$\frac{v'(l)}{u'(c)} = \left(1 - T'(z)\right) n$$

The uncompensated response of labor supply to the net of tax wage is:

$$\frac{\partial l^u}{\partial \left(1 - T'(z)\right) n} = \frac{u'(c) + l\left(1 - T'(z)\right) n u''(c)}{v''(l) - \left(1 - T'(z)\right)^2 n^2 u''(c)}$$

implying the following uncompensated elasticity:

$$\varepsilon^u = \frac{u'(c)/l + \frac{v'(l)^2}{u'(c)^2} u''(c)}{v''(l) - \frac{v'(l)^2}{u'(c)^2} u''(c)}$$

The response of labor to income changes is given by:

$$\frac{\partial l}{\partial I} = \frac{\left(1 - T'(z)\right) n u''(c)}{v''(l) - \left(1 - T'(z)\right)^2 n^2 u''(c)}$$

Using the Slutsky equation (as we did in Section notes 1):

$$\begin{aligned}\frac{\partial l^c}{\partial\left(1-T'(z)\right)n} &= \frac{u'(c)+l\left(1-T'(z)\right)nu''(c)}{v''(l)-\left(1-T'(z)\right)^2 n^2 u''(c)} - \frac{l\left(1-T'(z)\right)nu''(c)}{v''(l)-\left(1-T'(z)\right)^2 n^2 u''(c)} \\ &= \frac{u'(c)}{v''(l)-\left(1-T'(z)\right)^2 n^2 u''(c)}\end{aligned}$$

Therefore:

$$\varepsilon^c = \frac{v'(l)/l}{v''(l)-\left(1-T'(z)\right)^2 n^2 u''(c)}$$

Optimal Tax Everything is similar to the previous case except for the fact that now we cannot replace the variable $c(n)$ in the resource constraint using the definition of indirect utility. We will define consumption as an expenditure function $\tilde{c}(\tilde{u}(n), z(n), n)$ and implicitly differentiate it wrt to $\tilde{u}(n)$ and $z(n)$. Start from the definition of indirect utility:

$$\tilde{u}(n) = u(\tilde{c}(n)) - v(z^*(n)/n)$$

It follows that the following two conditions will hold at the optimum:

$$d\tilde{u}(n) = u'(\tilde{c}(n))\, d\tilde{c}(n)$$

$$0 = u'(\tilde{c}(n))\, d\tilde{c}(n) - \frac{1}{n} v'(z^*(n)/n)\, dz^*(n)$$

Rearranging:

$$\frac{d\tilde{c}(n)}{d\tilde{u}(n)} = \frac{1}{u'(\tilde{c}(n))}$$

$$\frac{d\tilde{c}(n)}{dz^*(n)} = \frac{v'(z^*(n)/n)}{nu'(\tilde{c}(n))}$$

The Hamiltonian for the problem is:

$$H = \left[G(u(n)) + \lambda\left(z(n) - \tilde{c}(\tilde{u}(n), z(n), n)\right)\right] f(n) + \mu(n)\frac{z(n)}{n^2} v'\left(\frac{z(n)}{n}\right)$$

and FOCs are:

$$\frac{\partial H}{\partial z(n)} = \lambda\left[1 - \frac{v'(z(n)/n)}{nu'(c(n))}\right] f(n) + \frac{\mu(n)}{n^2}\left[v'\left(\frac{z(n)}{n}\right) + \frac{z(n)}{n} v''\left(\frac{z(n)}{n}\right)\right] = 0$$

$$\frac{\partial H}{\partial u(n)} = \left[G'(u(n)) - \frac{\lambda}{u'(c^*(n))}\right] f(n) = -\mu'(n)$$

In order to find the equilibrium value of the multiplier, we can integrate the second FOC:

$$\mu(n) = \int_n^{\bar{n}} \left[G'(u(m)) - \frac{\lambda}{u'(c(m))}\right] f(m)\, dm$$

We exploit the definition of the two elasticities to write:

$$\left[v'\left(\frac{z(n)}{n}\right)+\frac{z(n)}{n}v''\left(\frac{z(n)}{n}\right)\right] = v'\left(\frac{z(n)}{n}\right)\left[1+\frac{z(n)}{n}\frac{v''\left(\frac{z(n)}{n}\right)}{v'\left(\frac{z(n)}{n}\right)}\right]$$
$$= v'\left(\frac{z(n)}{n}\right)\left(\frac{1+\varepsilon^{u}}{\varepsilon^{c}}\right)$$

The optimal tax formula will then become:

$$\frac{T'(z(n))}{1-T'(z(n))}=\left(\frac{1+\varepsilon^{u}}{\varepsilon^{c}}\right)\frac{\eta(n)}{nf(n)} \tag{5}$$

where $\eta(n)=\frac{u'(c(n))\mu(n)}{\lambda}$.

5 Pareto Efficient Taxes

We now ask the question of whether a tax system $T_0(z)$ in place is Pareto-optimal, meaning that there exists no feasible adjustment in the tax schedule such that all individuals in the economy are weakly better off.

We can characterize the Pareto frontier of the previous problem by solving the following:

$$\max\int_{\underline{n}}^{\bar{n}} u(n)\,\psi(n)\,dn$$

s.t.

$$u(c(n))-h(z(n)/n)\geq u(c(n'))-h(z(n')/n)\ \ \forall n,n'$$

$$\int_{\underline{n}}^{\bar{n}}[z(n)-c(n)]\,f(n)\,dn\geq E$$

By varying the social marginal welfare weights, we can trace out every point on the Pareto frontier. However, there might be points on the Pareto frontier that can be improved upon increasing the utility of all the agents in the economy.

Werning (2007) develops a test for the Pareto optimality of a tax schedule. The first important result of the paper is the following:

Proposition 1: A tax code fails to be constrained Pareto optimal *if and only if* there exists a feasible tax reform that (weakly) reduces taxes at all incomes.[1]

Proof: *(if)* suppose we weakly reduce taxes all over the entire economy, then every individual is at least as well off.

(only if) suppose there exists a Pareto improving feasible tax reform $T_1(z)$. Then we have:

$$\begin{aligned} U(z_1(n)-T_1(z_1(n)),z_1(n),n) &\geq U(z_o(n)-T_0(z_0(n)),z_0(n),n)\\ &\geq U(z_1(n)-T_0(z_1(n)),z_1(n),n)\end{aligned}$$

where the first inequality comes from the assumption of Pareto-improvement and the second from the assumption that under $T_0(z)$ the agent truthfully reveals her type and chooses $z_0(n)$. The chain of inequalities implies that $T_1(z_1(n))\leq T_0(z_1(n))$ for every n.

[1]Feasible means that it satisfies the resource constraint.

Proposition 1 implies that since the resource constraint is satisfied and both tax systems raise revenues at least equal to E, a Pareto improvement can only occur through a tax reduction that does not generate a drop in revenues. This can be interpreted as a Laffer effect: although the government lowers taxes, the behavioral response (increase in labor supply) is strong enough to more than compensate the revenue loss.

6 A Test of the Pareto Optimality of the Tax Schedule

In order to implement the test, Werning takes a dual approach to the optimal taxation problem that we studied in the previous paragraphs. We rewrite the problem such that instead of maximizing the social welfare function, the government maximizes the resources to provide a minimum level $\bar{v}(n)$ of indirect utility to every agent in the economy. We write the problem as follows:

$$\max_{u(n),z(n)} \int_{\underline{n}}^{\bar{n}} \left(z(n) - \tilde{c}(v(n), z(n), n)\right) f(n)\, dn$$

s.t.

$$\frac{dv(n)}{dn} = U_n\left(\tilde{c}(v(n), z(n), n), z(n), n\right) \tag{6}$$

$$v(n) \geq \bar{v}(n) \ \ \forall n \tag{7}$$

Notice that $\tilde{c}(v(n), z(n), n)$ is the expenditure function that we introduced to study optimal taxes with income effects. The problem would also have a monotonicity constraint that we relax for the moment, as we usually do. Notice that by changing the levels of $\bar{v}(n)$ we can characterize the entire Pareto frontier.[2]

We solve the problem with a Lagrangian by attaching multiplier $\psi(n) f(n)$ to (7) and $\mu(n)$ to the local incentive constraint:

$$\begin{aligned} L &= \int_{\underline{n}}^{\bar{n}} \left(z(n) - \tilde{c}(v(n), z(n), n)\right) f(n)\, dn + \int_{\underline{n}}^{\bar{n}} \psi(n) v(n) f(n)\, dn \\ &\quad + \int_{\underline{n}}^{\bar{n}} \mu(n) v'(n)\, dn - \int_{\underline{n}}^{\bar{n}} \mu(n) U_n\left(\tilde{c}(v(n), z(n), n), z(n), n\right) dn \end{aligned}$$

Notice that this is identical to the Lagrangian that we would obtain in a classical optimal tax problem with welfare weights $\psi(n) f(n)$.[3] We integrate $\int_{\underline{n}}^{\bar{n}} \mu(n) u'(n)\, dn$ by parts to obtain:

[2] The first order conditions for this problem will be sufficient. We can rewrite the problem in terms of $\tilde{u}(n) = u(c(n))$ and $\tilde{h}(n) = h(z(n)/n)$ so that the objective function becomes $nh^{-1}\left(\tilde{h}(n)\right) - u^{-1}(\tilde{u}(n))$ and is concave in $\tilde{u}(n)$ and $\tilde{h}(n)$ that become the new control and state variables.

[3] The problem would be:

$$\int_{\underline{n}}^{\bar{n}} v(n) \psi(n) f(n)\, dn$$

s.t.

$$\frac{dv(n)}{dn} = U_n\left(\tilde{c}(v(n), z(n), n), z(n), n\right)$$

$$\int_{\underline{n}}^{\bar{n}} \left[z(n) - \tilde{c}(v(n), z(n), n)\right] f(n)\, dn \geq E$$

$$\int_{\underline{n}}^{\bar{n}} \mu(n) v'(n)\, dn = \mu(\bar{n}) v(\bar{n}) - \mu(\underline{n}) v(\underline{n}) - \int_{\underline{n}}^{\bar{n}} \mu'(n) v(n)\, dn$$

and rewrite the Lagrangian as follows:

$$\begin{aligned} L &= \int_{\underline{n}}^{\bar{n}} (z(n) - \tilde{c}(v(n), z(n), n)) f(n)\, dn + \int_{\underline{n}}^{\bar{n}} \psi(n) v(n) f(n)\, dn \\ &\quad + \mu(\bar{n}) v(\bar{n}) - \mu(\underline{n}) v(\underline{n}) - \int_{\underline{n}}^{\bar{n}} \mu'(n) v(n)\, dn - \int_{\underline{n}}^{\bar{n}} \mu(n) U_n(\tilde{c}(v(n), z(n), n), z(n), n)\, dn \end{aligned}$$

The FOC wrt $z(n)$ is:

$$\left(1 - \frac{d\tilde{c}(v(n), z(n), n)}{dz(n)}\right) f(n) - \mu(n)\left[U_{nc}(n) \frac{d\tilde{c}(v(n), z(n), n)}{dz(n)} + U_{nz}(n)\right] = 0 \tag{8}$$

and the FOC wrt $v(n)$ is:

$$-\frac{d\tilde{c}(v(n), z(n), n)}{dv(n)} f(n) - \mu'(n) - \mu(n) U_{nc}(n) \frac{d\tilde{c}(v(n), z(n), n)}{dv(n)} + \psi(n) f(n) = 0 \tag{9}$$

We know from the paragraph about optimal taxation with income effects that we can write:

$$\frac{d\tilde{c}(v(n), z(n), n)}{dz(n)} = -\frac{U_z(n)}{U_c(n)} = MRS(n)$$

Also, since $T'(n) = 1 - MRS(n)$ (see the discussion about wedges) we have that:

$$1 - \frac{d\tilde{c}(v(n), z(n), n)}{dz(n)} = 1 - MRS(n) = T'(n)$$

The term in square brackets in equation (8) can be written as follows:

$$\begin{aligned} U_{nc}(n) \frac{d\tilde{c}(v(n), z(n), n)}{dz(n)} + U_{nz}(n) &= -U_{nc}(n) \frac{U_z(n)}{U_c(n)} + U_{nz}(n) \\ &= U_c(n) \frac{U_{zn}(n) U_c(n) - U_{cn}(n) U_z(n)}{U_c(n)^2} \\ &= -U_c(n) \frac{\partial}{\partial n}\left[-\frac{U_z(n)}{U_c(n)}\right] \\ &= -U_c(n) \frac{\partial MRS(n)}{\partial n} \end{aligned}$$

Therefore, euqation (8) becomes:

$$T'(n) f(n) = -\mu(n) U_c(n) \frac{\partial MRS(n)}{\partial n}$$

Using $MRS(n) = 1 - T'(n)$, we can rewrite the condition as:

$$\frac{T'(n)}{1 - T'(n)} f(n) = -\mu(n) U_c(n) \frac{\partial \log MRS(n)}{\partial n}$$

Now, we move to the second FOC. We know from before that:

$$\frac{d\tilde{c}\left(v\left(n\right),z\left(n\right),n\right)}{dv\left(n\right)}=\frac{1}{U_c\left(n\right)}$$

We can therefore rewrite (9) as:

$$-\frac{f\left(n\right)}{U_c\left(n\right)}-\mu'\left(n\right)-\mu\left(n\right)\frac{U_{nc}\left(n\right)}{U_c\left(n\right)}+\psi\left(n\right)f\left(n\right)=0$$

Any Pareto Efficient allocation must satisfy (7) and provide at least utility $v\left(\bar{n}\right)$ to every agent n. By the complementarity-slackness condition, this is equivalent to ask that $\psi\left(n\right)f\left(n\right)\geq 0$, which is that the multipliers associated to the constraints are never negative. We rewrite the FOC imposing the following inequality:

$$-U_c\mu'\left(n\right)-\mu\left(n\right)U_{nc}\left(n\right)\leq f\left(n\right) \tag{10}$$

If we change variables and define $\hat{\mu}\left(n\right)\equiv U_c\left(n\right)\mu\left(n\right)$, we have:

$$\hat{\mu}'\left(n\right)=U_c\left(n\right)\mu'\left(n\right)+\mu\left(n\right)\left[U_{cn}\left(n\right)+U_{cc}\left(n\right)c'\left(n\right)+U_{cz}\left(n\right)z'\left(n\right)\right]$$

Substituting into (10) we find:

$$-\hat{\mu}'\left(n\right)+\hat{\mu}\left(n\right)\frac{U_{cc}\left(n\right)c'\left(n\right)+U_{cz}\left(n\right)}{U_c\left(n\right)}\leq f\left(n\right)$$

The local incentive constraint of the agent (FOC for optimal reporting) implies that $c'\left(n\right)/z'\left(n\right)=-U_z\left(n\right)/U_c\left(n\right)$. It follows that:

$$\begin{aligned}\frac{U_{cc}\left(n\right)c'\left(n\right)+U_{cz}\left(n\right)}{U_c\left(n\right)} &= \frac{U_{cc}\left(n\right)\frac{c'\left(n\right)}{z'\left(n\right)}+U_{cz}\left(n\right)}{U_c\left(n\right)}z'\left(n\right)\\ &= \frac{-U_{cc}\left(n\right)U_z\left(n\right)+U_{cz}\left(n\right)U_c\left(n\right)}{U_c\left(n\right)^2}z'\left(n\right)\\ &= -\frac{\partial}{\partial c}\left[-\frac{U_z\left(n\right)}{U_c\left(n\right)}\right]z'\left(n\right)\\ &= -\frac{\partial MRS\left(n\right)}{\partial c}z'\left(n\right)\end{aligned}$$

We finally establish two conditions for Pareto efficiency:

$$\frac{T'\left(n\right)}{1+T'\left(n\right)}f\left(n\right)=-\hat{\mu}\left(n\right)\frac{\partial\log MRS\left(n\right)}{\partial n} \tag{11}$$

$$-\hat{\mu}'\left(n\right)-\hat{\mu}\left(n\right)\frac{\partial MRS\left(n\right)}{\partial c}z'\left(n\right)\leq f\left(n\right) \tag{12}$$

Using the tax schedule in place, the preferences and the skill distribution we can derive the $\hat{\mu}$ from equation (11). We can then use equation (12) to test for the Pareto efficiency of the tax schedule.

Applying the Test and Interpreting the Conditions Suppose the agent has preferences $U(c, z, n) = c - \frac{1}{\gamma}(z/n)^\gamma$ with elasticity $\varepsilon = 1/(\gamma - 1)$. The FOCs of the dual problem read:

$$\mu'(n) = \psi(n) f(n) - f(n) \tag{13}$$

$$\frac{T'(n)}{1 - T'(n)} f(n) = \frac{\mu(n)}{n} \frac{1 + \varepsilon}{\varepsilon} \tag{14}$$

The tax schedule is Pareto-optimal if and only if $\psi(n) f(n) \geq 0$, which implies $-\mu'(n) \leq f(n)$. This inequality is the same as the one derived in (12) since $U_c(n) = 1$ and $\partial MRS(n)/\partial c = 0$. Equation (14) is the same as (11) when we notice that $\partial MRS(n)/\partial n = (1 + 1/\varepsilon)/n$.[4]

Suppose that the marginal tax rate is linear and equal to τ, when we put the two conditions together we get:

$$-\frac{\varepsilon}{1+\varepsilon} \frac{\partial}{\partial n} \left[\frac{\tau}{1-\tau} n f(n) \right] = -\mu'(n) \leq f(n) \;\; \forall n$$

Taking the derivative wrt n the condition becomes:

$$-\frac{\varepsilon}{1+\varepsilon} \frac{\tau}{1-\tau} \left[-1 - \frac{n f'(n)}{f(n)} \right] \leq 1 \; \forall n \tag{15}$$

First, the condition in (15) shows that for any τ and ε there exists a set of $f(n)$ such that τ is Pareto efficient and a set of $f(n)$ such that it is not Pareto efficient. At the same time, for any ε and $f(n)$ we can find flat tax schedules τ that are efficient and set of τs that are inefficient. It follows that it is crucial to know the distribution of skills. The test can also be written in terms of income distributions that are easier to infer from the data. Higher ε makes the condition harder to be satisfied: when individuals are reacting more to changes in taxes, a tax reduction is more likely to lead to a Pareto improvement. When taxes are locally lowered at some n, the individuals below n will tend to increase their labor supply and individuals above will reduce the labor supply. The term $nf'(n)/f(n)$ measures the elasticity of the skill distribution and captures how fast the skill distribution is decreasing at some n. Highly negative elasticity of skill distribution at n means that the distribution decreases fast and that the mass of individuals below n is significantly larger than above n, implying a local Laffer effect from the increase in labor supply of individuals below n. In other words, by locally decreasing taxes at n the government can increase revenues by incentivizing the labor supply of the large mass of individuals below n. For this reason, when the elasticity of the skill distribution is highly negative the test is harder to pass.[5]

References

[1] Diamond, Peter (1998), "Optimal Income Taxation: An Example with a U-Shaped Pattern of Optimal Marginal Tax Rates", American Economic Review, 88, 83–95.

[2] Salaniè, Bernard (2002), "The Economics of Taxation", The MIT Press

[3] Scheuer, Florian (2016), Leture Notes (find at `http://web.stanford.edu/~scheuer/2016_class3.pdf`)

[4] Werning, Ivan (2007), "Pareto Efficient Income Taxation", Working Paper

[4] Notice that $MRS(n) = z(n)^{\gamma-1} n^{-\gamma}$
Therefore:

$$\frac{\partial \log MRS(n)}{\partial n} = \frac{\partial MRS(n)}{\partial n} \frac{1}{MRS(n)} = -\frac{\gamma}{n}$$

where $\gamma = 1 + 1/\varepsilon$.

[5] In the second part of the sequence you will study how to run the same test in the inequality deflator framework using the concept of fiscal externality.